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L_p Markov–Bernstein Inequalities on All Arcs of the Circle

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Let $0<p<\infty$ and $0\leqslant\alpha<\beta\leqslant 2\pi$. We prove that for $n\geqslant 1$ and trigonometric polynomials s_n of degree $\leqslant n$, we have

$$\int_\alpha^\beta |s_n'(\theta)|^p \left[\frac{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\right|\left|\sin\left(\frac{\theta-\beta}{2}\right)\right|+\left(\frac{\beta-\alpha}{n}\right)^2}{\left(\cos\frac{\theta-\frac{\alpha+\beta}{2}}{2}\right)^2+\left(\frac{1}{n}\right)^2}\right]^{p/2} d\theta$$

$$\leqslant cn^p \int_\alpha^\beta |s_n(\theta)|^p\, d\theta,$$

where c is independent of α, β, n, s_n. The essential feature is the uniformity in $[\alpha,\beta]$ of the estimate and the fact that as $[\alpha,\beta]$ approaches $[0,2\pi]$, we recover the L_p Markov inequality. The result may be viewed as the complete L_p form of Videnskii's inequalities, improving earlier work of the second author.

1. INTRODUCTION AND RESULTS

The classical Markov–Bernstein inequality for trigonometric polynomials

$$s_n(\theta) := \sum_{j=0}^{n} (c_j \cos j\theta + d_j \sin j\theta)$$

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of degree $\leqslant n$ is

$$\|s_n'\|_{L_\infty[0,\,2\pi]} \leqslant n\,\|s_n\|_{L_\infty[0,\,2\pi]}.$$

The same factor n occurs in the L_p analogue. See [1] or [3]. In the 1950s V. S. Videnskii generalized the L_∞ inequality to the case where the interval over which the norm is taken is shorter than the period [1, pp. 242–245]: let $0<\omega<\pi$. Then there is the sharp inequality

$$|s_n'(\theta)|\left[1-\left(\frac{\cos\omega/2}{\cos\theta/2}\right)^2\right]^{1/2} \leqslant n\,\|s_n\|_{L_\infty[-\omega,\,\omega]}, \qquad \theta\in[-\omega,\omega].$$

This implies that

$$\sup_{\theta\in[-\pi,\pi]} |s_n'(\theta)|\left[\left|\sin\left(\frac{\theta-\omega}{2}\right)\right|\left|\sin\left(\frac{\theta+\omega}{2}\right)\right|\right]^{1/2} \leqslant n\,\|s_n\|_{L_\infty[-\omega,\,\omega]}$$

and for $n\geqslant n_0(\omega)$, gives rise to the sharp Markov inequality

$$\|s_n'\|_{L_\infty[-\omega,\,\omega]} \leqslant 2n^2\cot\frac{\omega}{2}\,\|s_n\|_{L_\infty[-\omega,\,\omega]}. \tag{1}$$

What are the L_p analogues? This question arose originally in connection with large sieve inequalities [7], on subarcs of the circle. In an earlier paper, the second author proved the following result:

THEOREM 1.1. *Let $0<p<\infty$ and $0\leqslant\alpha<\beta\leqslant2\pi$. Then for $n\geqslant1$ and trigonometric polynomials s_n of degree $\leqslant n$,*

$$\int_\alpha^\beta |s_n'(\theta)|^p\left[\left|\sin\left(\frac{\theta-\alpha}{2}\right)\right|\left|\sin\left(\frac{\theta-\beta}{2}\right)\right|+\left(\frac{\beta-\alpha}{n}\right)^2\right]^{p/2} d\theta \tag{2}$$

$$\leqslant Cn^p\int_\alpha^\beta |s_n(\theta)|^p\,d\theta.$$

Here C is independent of α, β, n, s_n.

This inequality confirmed a conjecture of Erdelyi [4]. Theorem 1.1 was deduced from an analogous inequality for algebraic polynomials.

While Theorem 1.1 is almost certainly sharp with respect to the growth in n when $[\alpha,\beta]$ is a fixed proper subinterval of $(0,\pi)$, and most especially when $[\alpha,\beta]$ is small, it is not sharp when $[\alpha,\beta]$ approaches $[0,2\pi]$. For example, Theorem 1.1 gives

$$\int_0^{2\pi} |s_n'(\theta)|^p\left[\left(\sin\frac{\theta}{2}\right)^2+\left(\frac{2\pi}{n}\right)^2\right]^{p/2} d\theta \leqslant Cn^p\int_\alpha^\beta |s_n(\theta)|^p\,d\theta,$$

while the correct Markov inequality is (with $C=1$),

$$\int_0^{2\pi} |s_n'(\theta)|^p \, d\theta \leqslant Cn^p \int_0^{2\pi} |s_n(\theta)|^p \, d\theta. \tag{3}$$

It is possible to derive this by two applications of (2) (on different intervals) and then by using 2π-periodicity of the integrand. However, for general $[\alpha, \beta] \subset [0, 2\pi]$, we are not able to use 2π-periodicity, so for $\beta-\alpha$ close to 2π, it seems that we cannot obtain the sharp result from (2). In this paper, we establish an improvement of Theorem 1.1 which does yield (3) and is almost certainly sharp for $[\alpha, \beta]$ close to $[0, 2\pi]$. In particular, we prove:

THEOREM 1.2. *Let $0<p<\infty$ and $0 \leqslant \alpha < \beta \leqslant 2\pi$. Then for $n \geqslant 1$ and trigonometric polynomials s_n of degree $\leqslant n$,*

$$\int_\alpha^\beta |s_n'(\theta)|^p \left[\frac{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\right| \left|\sin\left(\frac{\theta-\beta}{2}\right)\right| + \left(\frac{\beta-\alpha}{n}\right)^2}{\left(\cos\frac{\theta-\frac{\alpha+\beta}{2}}{2}\right)^2 + \left(\frac{1}{n}\right)^2} \right]^{p/2} d\theta \tag{4}$$

$$\leqslant Cn^p \int_\alpha^\beta |s_n(\theta)|^p \, d\theta.$$

Here C is independent of α, β, n, s_n.

For example, if we take our interval to be $[-\omega, \omega]$, where $0<\omega<\pi$, we may reformulate the above inequality as

$$\int_{-\omega}^{\omega} |s_n'(\theta)|^p \left[\frac{\left|\sin\left(\frac{\theta-\omega}{2}\right)\right| \left|\sin\left(\frac{\theta+\omega}{2}\right)\right| + \left(\frac{2\omega}{n}\right)^2}{\left(\cos\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2} \right]^{p/2} d\theta \tag{5}$$

$$\leqslant Cn^p \int_{-\omega}^{\omega} |s_n(\theta)|^p \, d\theta,$$

with C independent of ω, n, s_n, or equivalently,

$$\int_{-\omega}^{\omega} |s_n'(\theta)|^p \left[\frac{\left(\cos\frac{\theta}{2}\right)^2 - \left(\cos\frac{\omega}{2}\right)^2 + \left(\frac{2\omega}{n}\right)^2}{\left(\cos\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2} \right]^{p/2} d\theta \leqslant Cn^p \int_{-\omega}^{\omega} |s_n(\theta)|^p \, d\theta. \tag{6}$$

As $\omega \to \pi$, we recover the Markov inequality (3). Note that also as ω becomes small, (5) reduces to

$$\int_{-\omega}^{\omega} |s_n'(\theta)|^p \left[\left| \sin\left(\frac{\theta-\omega}{2}\right)\right| \left| \sin\left(\frac{\theta+\omega}{2}\right)\right| + \left(\frac{2\omega}{n}\right)^2 \right]^{p/2} d\theta$$

$$\leqslant C n^p \int_{-\omega}^{\omega} |s_n(\theta)|^p \, d\theta,$$

which in turn implies the L_p Markov inequality

$$\int_{-\omega}^{\omega} |s_n'(\theta)|^p \, d\theta \leqslant C \left(\frac{n^2}{\omega}\right)^p \int_{-\omega}^{\omega} |s_n(\theta)|^p \, d\theta.$$

The latter is the L_p version of (1).

We shall deduce Theorem 1.2 from:

THEOREM 1.3. *Let $0 < p < \infty$ and $0 \leqslant \alpha < \beta \leqslant 2\pi$. Let*

$$\varepsilon_n(z) := \frac{1}{n} \left[\frac{|z - e^{i\alpha}|\,|z - e^{i\beta}| + \left(\frac{\beta-\alpha}{n}\right)^2}{|z + e^{i\frac{\alpha+\beta}{2}}|^2 + \left(\frac{1}{n}\right)^2} \right]^{1/2}. \tag{7}$$

Then for $n \geqslant 1$ and algebraic polynomials P of degree $\leqslant n$,

$$\int_{\alpha}^{\beta} |(P'\varepsilon_n)(e^{i\theta})|^p \, d\theta \leqslant C \int_{\alpha}^{\beta} |P(e^{i\theta})|^p \, d\theta. \tag{8}$$

Here C is independent of α, β, n, s_n.

Our method of proof uses Carleson measures much as in [8–10], but also uses ideas from [7] where large sieve inequalities were proved for subarcs of the circle. Despite the similarities in many of the proofs, especially to those in [10], we provide the details, for otherwise the proofs would be very difficult to follow. The chief difference to the proofs in [10] is due to the more delicate choice of ε_n, which substantially complicates the proofs in Section 3.

We shall prove Theorem 1.3 in Section 2, deferring some technical estimates. In Section 3, we present estimates involving the function ε_n and also estimate the norms of certain Carleson measures. In Section 4, we prove Theorem 1.2.

2. THE PROOF OF THEOREM 1.3

Throughout, $C, C_0, C_1, C_2, \ldots$ denote constants that are independent of α, β, ω, n and polynomials P of degree $\leqslant n$ or trigonometric polynomials s_n of degree $\leqslant n$. They may, however, depend on p. The same symbol does not necessarily denote the same constant in different occurrences. We shall prove Theorem 1.3 in several steps:

I. *Reduction to the Case* $0 < \alpha < \pi$; $\beta := 2\pi - \alpha$

After a rotation of the circle, we may assume that our arc $\{e^{i\theta} : \theta \in [\alpha, \beta]\}$ has the form

$$\Delta = \{e^{i\theta} : \theta \in [\alpha', 2\pi - \alpha']\},$$

where $0 \leqslant \alpha' < \pi$. Then Δ is symmetric about the real line, and this simplifies use of a conformal map below. Moreover, then

$$\beta - \alpha = 2(\pi - \alpha').$$

Dropping the prime, it suffices to consider $0 < \alpha < \pi$, and $\beta - \alpha$ replaced everywhere by $2(\pi - \alpha)$. Thus in the following we assume that

$$\Delta = \{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha]\}; \tag{9}$$

$$R(z) = (z - e^{i\alpha})(z - e^{-i\alpha}) = z^2 - 2z\cos\alpha + 1. \tag{10}$$

Since then $\frac{\alpha+\beta}{2} = \pi$, we may take for $z = e^{i\theta}$ (dropping the subscript n from ε_n in (7) and a factor of 2 in $\pi - \alpha$),

$$\varepsilon(z) = \frac{1}{n}\left[\frac{|R(z)| + \left(\frac{\pi-\alpha}{n}\right)^2}{|z-1|^2 + \left(\frac{1}{n}\right)^2}\right]^{1/2} \tag{11}$$

$$= \frac{1}{n}\left[\frac{4\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\right| + \left(\frac{\pi-\alpha}{n}\right)^2}{4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2}\right]^{1/2}.$$

We can now begin the main part of the proof:

II. *Pointwise Estimates for $P'(z)$ when $p \geqslant 1$*

By Cauchy's integral formula for derivatives (or by Cauchy's estimates),

$$|P'(z)| = \left| \frac{1}{2\pi i} \int_{|t-z| = \varepsilon(z)/100} \frac{P(t)}{(t-z)^2} \, dt \right|$$
$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta} \right) \right| d\theta \bigg/ \left(\frac{\varepsilon(z)}{100} \right).$$

Then Hölder's inequality gives

$$|P'(z)| \, \varepsilon(z) \leqslant 100 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta} \right) \right|^p d\theta \right)^{1/p}$$
$$\Rightarrow (|P'(z)| \, \varepsilon(z))^p \leqslant 100^p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta} \right) \right|^p d\theta.$$

III. *Pointwise Estimates for $P'(z)$ when $p < 1$*

We follow ideas in [9, 10]. Suppose first that P has no zeros inside or on the circle $\gamma := \{t : |t-z| = \frac{\varepsilon(z)}{100}\}$. Then we can choose a single valued branch of P^p there, with the properties

$$\frac{d}{dt} P(t)^p_{|t=z} = pP(z)^p \frac{P'(z)}{P(z)}$$

and

$$|P^p(t)| = |P(t)|^p.$$

Then by Cauchy's integral formula for derivatives,

$$p \, |P'(z)| \, |P(z)|^{p-1} = \left| \frac{1}{2\pi i} \int_{|t-z| = \frac{\varepsilon(z)}{100}} \frac{P^p(t)}{(t-z)^2} \, dt \right|$$
$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta} \right) \right|^p d\theta \bigg/ \left(\frac{\varepsilon(z)}{100} \right).$$

Since also (by Cauchy or by subharmonicity)

$$|P(z)|^p \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta} \right) \right|^p d\theta$$

and since $1-p>0$, we deduce that

$$p\,|P'(z)|\,\varepsilon(z)\leqslant 100\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta\right)^{1/p}$$

$$\Rightarrow (|P'(z)|\,\varepsilon(z))^{p}\leqslant\left(\frac{100}{p}\right)^{p}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta.$$

Now suppose that P has zeros inside γ. We may assume that it does not have zeros on γ (if necessary change $\varepsilon(z)$ a little and then use continuity). Let $B(z)$ be the Blaschke product formed from the zeros of P inside γ. This is the usual Blaschke product for the unit circle, but scaled to γ so that $|B|=1$ on γ. Then the above argument applied to (P/B) gives

$$(|(P/B)'\,(z)|\,\varepsilon(z))^{p}\leqslant\left(\frac{100}{p}\right)^{p}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta.$$

Moreover, as above

$$|P/B(z)|^{p}\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta,$$

while Cauchy's estimates give

$$|B'(z)|\leqslant\frac{100}{\varepsilon(z)}.$$

Then these last three estimates give

$$|P'(z)|^{p}\,\varepsilon(z)^{p}\leqslant(|(P/B)'\,(z)\,B(z)|+|P/B(z)|\,|B'(z)|)^{p}\,\varepsilon(z)^{p}$$

$$\leqslant\left\{\left(\frac{200}{p}\right)^{p}+200^{p}\right\}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta\right].$$

In summary, the last two steps give for all $p>0$,

$$|P'\varepsilon|^{p}\,(z)\leqslant C_0\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}\,e^{i\theta}\right)\right|^{p}d\theta, \tag{12}$$

where

$$C_0:=200^{p}(1+p^{-p}).$$

IV. *Integrate the Pointwise Estimates*

We obtain by integration of (12) that

$$\int_{\alpha}^{2\pi-\alpha}|(P'\varepsilon)(e^{i\theta})|^{p}\,d\theta\leqslant C_0\int|P(z)|^{p}\,d\sigma, \tag{13}$$

where the measure σ is defined by

(14) $$\int f\, d\sigma := \int_{\alpha}^{2\pi-\alpha} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{is} + \frac{\varepsilon(e^{is})}{100}\, e^{i\theta} \right) d\theta \right] ds.$$

We now wish to pass from the right-hand side of (13) to all estimate over the whole unit circle. This passage would be permitted by a result of Carleson, provided P is analytic off the unit circle and provided it has suitable behaviour at ∞. To take care of the fact that it does not have the correct behaviour at ∞, we need a conformal map:

V. *The Conformal Map* Ψ *of* $\mathbb{C}\setminus\Delta$ *onto* $\{w: |w|>1\}$

This is given by

$$\Psi(z) = \frac{1}{2\cos\alpha/2}\,[z+1+\sqrt{R(z)}],$$

where the branch of $\sqrt{R(z)}$ is chosen so that it is analytic off Δ and behaves like $z(1+o(1))$ as $z\to\infty$. Note that $\sqrt{R(z)}$ and hence $\Psi(z)$ have well-defined boundary values (both nontangential and tangential) as z approaches Δ from inside or outside the unit circle, except at $z=e^{\pm i\alpha}$. We denote the boundary values from inside by $\sqrt{R(z)}_+$ and $\Psi(z)_+$ and from outside by $\sqrt{R(z)}_-$ and $\Psi(z)_-$. We also set (unless otherwise specified)

$$\Psi(z) := \Psi(z)_-, \qquad z\in\Delta\setminus\{e^{i\alpha}, e^{-i\alpha}\}.$$

See [6] for a detailed discussion and derivation of this conformal map. Let

(15) $$\ell := \text{least positive integer} > \frac{1}{p}.$$

In Lemma 3.2 we shall show that there is a constant C_1 (independent of α, β, n) such that

$$a\in\Delta \quad \text{and} \quad |z-a| \leqslant \frac{\varepsilon(a)}{100} \Rightarrow |\Psi(z)|^{n+\ell} \leqslant C_1.$$

Then we deduce from (13) that

(16) $$\int_{\alpha}^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p\, d\theta \leqslant C_1^p C_0 \int \left| \frac{P}{\Psi^{n+\ell}} \right|^p d\sigma.$$

Since the form of Carleson's inequality that we use involves functions analytic inside the unit ball, we now split σ into its parts with support inside and outside the unit circle: for measurable S, let

$$\sigma^+(S) := \sigma(S \cap \{z : |z| < 1\});$$
$$\sigma^-(S) := \sigma(S \cap \{z : |z| > 1\}). \tag{17}$$

Moreover, we need to "reflect σ^- through the unit circle": let

$$\sigma^\#(S) := \sigma^-\left(\frac{1}{S}\right) := \sigma^-\left(\left\{\frac{1}{t} : t \in S\right\}\right). \tag{18}$$

Then since the unit circle Γ has $\sigma(\Gamma) = 0$, (16) becomes

$$\int_\alpha^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \tag{19}$$
$$\leqslant C_1^p C_0 \left(\int \left|\frac{P}{\Psi^{n+\ell}}\right|^p (t) \, d\sigma^+(t) + \int \left|\frac{P}{\Psi^{n+\ell}}\right|^p \left(\frac{1}{t}\right) d\sigma^\#(t)\right).$$

We next focus on handling the first integral in the last right-hand side:

VI. *Estimate the Integral Involving* σ^+

We are now ready to apply Carleson's result. Recall that a positive Borel measure μ with support inside the unit ball is called a *Carleson measure* if there exists $A > 0$ such that for every $0 < h < 1$ and every sector

$$S := \{re^{i\theta} : r \in [1-h, 1]; |\theta - \theta_0| \leqslant h\}$$

we have

$$\mu(S) \leqslant Ah.$$

The smallest such A is called the Carleson norm of μ and denoted $N(\mu)$. See [5] for an introduction. One feature of such a measure is the inequality

$$\int |f|^p \, d\mu \leqslant C_2 N(\mu) \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{20}$$

valid for every function f in the Hardy p space on the unit ball. Here C_2 depends only on p. See [5, p. 238] and also [5, pp. 31–63].

Applying this to $P/\Psi^{n+\ell}$ gives

$$\int \left|\frac{P}{\Psi^{n+\ell}}\right|^p d\sigma^+ \leqslant C_2 N(\sigma^+) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{i\theta})\right|^p d\theta. \tag{21}$$

VII. *Estimate the Integral Involving* $\sigma^{\#}$

Suppose that P has degree $\nu \leqslant n$. As $\Psi(z)/z$ has a finite nonzero limit as $z \to \infty$, $P(z)/\Psi(z)^{\nu}$ has a finite nonzero limit as $z \to \infty$. Then $h(t) := p(\frac{1}{t})/\Psi(\frac{1}{t})^{n+\ell}$ has zeros in $|t| < 1$ corresponding only to zeros of $P(z)$ in $|z| > 1$ and a zero of multiplicity $n+\ell-\nu$ at $t = 0$, corresponding to the zero of $P(z)/\Psi(z)^{n+\ell}$ at $z = \infty$. Then we may apply Carleson's inequality (20) to h. The consequence is that

$$\int \left|\frac{P}{\Psi^{n+\ell}}\right|^p \left(\frac{1}{t}\right) d\sigma^{\#}(t) \leqslant C_2 N(\sigma^{\#}) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{-i\theta})\right|^p d\theta.$$

Combined with (19) and (21), this gives

(22)

$$\int_{\alpha}^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \leqslant C_0 C_1^p C_2 (N(\sigma^+) + N(\sigma^{\#})) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{i\theta})\right|^p d\theta.$$

VIII. *Pass from the Whole Unit Circle to* Δ *when* $p > 1$

Let Γ denote the whole unit circle, and let $|dt|$ denote arclength on Γ. In Step VIII of the proof of Theorem 1.2 in [10], we established an estimate of the form

$$\int_{\Gamma \setminus \Delta} |g(t)|^p \, |dt| \leqslant C_3 \left(\int_{\Delta} |g_+(t)|^p \, |dt| + |g_-(t)|^p \, |dt| \right), \tag{23}$$

valid for all functions g analytic in $\mathbb{C} \setminus \Delta$, with limit 0 at ∞ and interior and exterior boundary values g_+ and g_- for which the right-hand side of (23) is finite. Here, C_3 depends only on p. We apply this to $g := P/\Psi^{n+\ell}$. Then as $\Psi_{\pm}$ have absolute value 1 on Δ, so that $|g_{\pm}| = |P|$ on Δ, we deduce that

$$\int_{\Gamma \setminus \Delta} |P(t)/\Psi(t)^{n+\ell}|^p \, |dt| \leqslant C_3 \int_{\Delta} |P(t)|^p \, |dt|$$

$$\Rightarrow \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{i\theta})\right|^p d\theta \leqslant \left(\int_{\alpha}^{2\pi-\alpha} |P(e^{i\theta})|^p \, d\theta \right) (1 + C_3).$$

Now (22) becomes

$$\int_{\alpha}^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \leqslant C_0 C_1^p C_2 (1 + C_3)(N(\sigma^+) + N(\sigma^{\#})) \int_{\alpha}^{2\pi-\alpha} |P(e^{i\theta})|^p \, d\theta. \tag{24}$$

IX. *Pass from the Whole Unit Circle to Δ when $p \leqslant 1$*

It is only here that we really need the choice (15) of ℓ. Let

$$q := \ell p (> 1).$$

Then we would like to apply (23) with p replaced by q and with

$$g := (P/\Psi^n)^{p/q}\, \Psi^{-1} = (P/\Psi^{n+\ell})^{p/q}. \tag{25}$$

The problem is that g does not in general possess the required properties. To circumvent this, we proceed as follows: first, we may assume that P has full degree n. For, if P has degree $< n$, we can add a term of the form δz^n, giving $P(z) + \delta z^n$, a polynomial of full degree n. Once (8) is proved for such P, we can then let $\delta \to 0+$.

So assume that P has degree n. Then P/Ψ^n is analytic in $\mathbb{C} \setminus \Delta$ and has a finite nonzero limit at ∞, and so is analytic at ∞. Now if all zeros of P lie on Δ, then we may define a single-valued branch of g of (25) in $\overline{\mathbb{C}} \setminus \Delta$. Then (23) with q replacing p gives as before

$$\int_{\Gamma \setminus \Delta} |g(t)|^q\, |dt| \leqslant C_3 \left(\int_\Delta |g_+(t)|^q\, |dt| + |g_-(t)|^q\, |dt| \right)$$

$$\Rightarrow \int_{\Gamma \setminus \Delta} |P/\Psi^{n+\ell}|^p\, |dt| \leqslant 2C_3 \int_\Delta |P(t)|^p\, |dt|$$

and then we obtain an estimate similar to (24). When P has zeros in $\mathbb{C} \setminus \Delta$, we adopt a standard procedure to "reflect" these out of $\mathbb{C} \setminus \Delta$. Write

$$P(z) = d \prod_{j=1}^{n} (z - z_j).$$

For each factor $z - z_j$ in P with $z_j \notin \Delta$, we define

$$b_j(z) := \begin{cases} (z - z_j) \Big/ \left(\dfrac{\Psi(z) - \Psi(z_j)}{1 - \overline{\Psi(z_j)}\, \Psi(z)} \right), & z \neq z_j, \\ (1 - |\Psi(z_j)|^2)/\Psi'(z_j), & z = z_j. \end{cases}$$

This is analytic in $\mathbb{C} \setminus \Delta$, does not have any zeros there, and moreover, since as $z \to \Delta$, $|\Psi(z)| \to 1$, we see that

$$|b_j(z)| = |z - z_j|, \qquad z \in \Delta; \qquad |b_j(z)| \geqslant |z - z_j|, \qquad z \in \mathbb{C} \setminus \Delta.$$

(Recall that we extended Ψ to Δ as an exterior boundary value.) We may now choose a branch of

$$g(z) := \left[d \left(\prod_{z_j \notin \Delta} b_j(z) \right) \left(\prod_{z_j \in \Delta} (z - z_j) \right) \Big/ \Psi(z)^n \right]^{p/q} \Big/ \Psi(z)$$

that is single valued and analytic in $\mathbb{C}\setminus\Delta$ and has limit 0 at ∞. Then as $\Psi_\pm$ have absolute value 1 on Δ, so that $|g_\pm|^q = |P|^p$ on Δ, we deduce from (23) that

$$\int_{\Gamma\setminus\Delta} |P(t)/\Psi(t)^{n+\ell}|^p \, |dt| \leqslant \int_{\Gamma\setminus\Delta} |g(t)|^q \, |dt|$$

$$\leqslant C_3 \int_\Delta (|g_+(t)|^q + |g_-(t)|^q) \, |dt| = 2C_3 \int_\Delta |P(t)|^p \, |dt|$$

and again we obtain an estimate similar to (24).

X. *Completion of the Proof*

We shall show in Lemma 3.3 that

$$N(\sigma^+) + N(\sigma^\#) \leqslant C_4. \tag{26}$$

Then (24) becomes

$$\int_\alpha^{2\pi-\alpha} |(P'\varepsilon_n)(e^{i\theta})|^p \, d\theta \leqslant C_5 \int_\alpha^{2\pi-\alpha} |P(e^{i\theta})|^p \, d\theta.$$

So we have (8) with a constant C_5 that depends only on the numerical constants C_j, $1 \leqslant j \leqslant 4$ that arise from

(a) the bound on the conformal map Ψ;

(b) Carleson's inequality (20);

(c) the norm of the Hilbert transform as an operator on $L_p(\Gamma)$ and the choice of ℓ;

(d) the upper bound on the Carleson norms of $\sigma^+$and $\sigma^\#$.

3. TECHNICAL ESTIMATES

Throughout we assume (9) to (11). Recall that

$$\begin{aligned} R(e^{i\theta}) &= (e^{i\theta} - e^{i\alpha})(e^{i\theta} - e^{-i\alpha}) \\ &= -4e^{i\theta} \sin\left(\frac{\theta-\alpha}{2}\right) \sin\left(\frac{\theta+\alpha}{2}\right) \\ &= -4e^{i\theta}\left(\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}\right) \\ &= -4e^{i\theta}\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\alpha}{2}\right). \end{aligned} \tag{27}$$

From this, we derive the following bounds, valid for $\theta \in [\alpha, 2\pi - \alpha]$:

$$|R(e^{i\theta})| \leqslant 4\left(\sin\frac{\theta}{2}\right)^2, \tag{28}$$

$$|R(e^{i\theta})| \leqslant 4\left(\cos\frac{\alpha}{2}\right)^2, \tag{29}$$

$$|R(e^{i\theta})| \leqslant 4\left|\sin\frac{\theta}{2}\right|\cos\frac{\alpha}{2}. \tag{30}$$

Our first lemma deals with properties of $\varepsilon(z)$ of (11),

$$\varepsilon(e^{i\theta}) = \varepsilon_n(e^{i\theta}) = \frac{1}{n}\left[\frac{4\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\right| + \left(\frac{\pi-\alpha}{n}\right)^2}{4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2}\right]^{1/2}.$$

Note that we drop the subscript n, as in the previous section, to simplify notation.

LEMMA 3.1. (a) *For $a \in \Delta$,*

$$|\varepsilon(e^{i\theta})| \leqslant 6\,\frac{\cos\frac{\alpha}{2}}{n}. \tag{31}$$

(b) *For $a, z \in \Delta$,*

$$|\varepsilon(z) - \varepsilon(a)| \leqslant 14\,|z-a|. \tag{32}$$

(c) *For $a, z \in \Delta$ such that $|z-a| \leqslant \frac{1}{28}\,\varepsilon(a)$, we have*

$$\frac{1}{2} \leqslant \frac{\varepsilon(z)}{\varepsilon(a)} \leqslant \frac{3}{2}. \tag{33}$$

(d) *Let $\theta \in [0, 2\pi]$ be given and let $s \in [0, 2\pi]$ satisfy*

$$|e^{is} - e^{i\theta}| \leqslant r < 2.$$

Then s belongs to a set of linear Lebesgue measure at most $2\pi r$.

Proof. We shall write

$$f(\theta) := |R(e^{i\theta})| + \left(\frac{\pi-\alpha}{n}\right)^2,$$

$$g(\theta) := 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2,$$

so that

$$\varepsilon(e^{i\theta}) = \frac{1}{n}\left(\frac{f(\theta)}{g(\theta)}\right)^{1/2}.$$

(a) It follows from (28) that

(34) $$f(\theta) \leqslant 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2 \leqslant \pi^2 g(\theta),$$

so that

$$\varepsilon(e^{i\theta}) \leqslant \frac{\pi}{n}.$$

Also, from the inequality

(35) $$\frac{\pi-\alpha}{\pi} \leqslant \cos\frac{\alpha}{2} = \sin\left(\frac{\pi-\alpha}{2}\right) \leqslant \frac{\pi-\alpha}{2},$$

and from (29), we obtain

$$\varepsilon(e^{i\theta}) \leqslant \frac{(4+\pi^2)^{1/2}}{n} \frac{\cos\frac{\alpha}{2}}{\left|\sin\frac{\theta}{2}\right|} \leqslant \frac{4}{n} \frac{\cos\alpha/2}{\sin\alpha/2}.$$

Then the two bounds on ε give

$$\frac{\varepsilon(e^{i\theta})}{\cos\frac{\alpha}{2}} \leqslant \frac{4}{n} \min\left\{\frac{1}{\cos\frac{\alpha}{2}}, \frac{1}{\sin\frac{\alpha}{2}}\right\} \leqslant \frac{6}{n}.$$

(b) Write $z=e^{i\theta}$; $a=e^{is}$. We shall assume, as we may, that

(36) $$\left|\sin\frac{s}{2}\right| \geqslant \left|\sin\frac{\theta}{2}\right|$$

or, equivalently, that s is closer to π than θ. Note from the definition of f, g, and (27) that

$$f(\theta)=g(\theta)+c,$$

where

$$c=-4\left(\sin\frac{\alpha}{2}\right)^2+\frac{(\pi-\alpha)^2-1}{n^2}.$$

Then

$$\varepsilon(e^{i\theta})=\frac{1}{n}\left(1+\frac{c}{g(\theta)}\right)^{1/2},$$

so

$$n[\varepsilon(e^{i\theta})-\varepsilon(e^{is})]=\frac{\left(1+\frac{c}{g(\theta)}\right)-\left(1+\frac{c}{g(s)}\right)}{\left(1+\frac{c}{g(\theta)}\right)^{1/2}+\left(1+\frac{c}{g(s)}\right)^{1/2}}$$

$$=\frac{c[g(s)-g(\theta)]}{g(\theta)\,g(s)\left[\left(1+\frac{c}{g(\theta)}\right)^{1/2}+\left(1+\frac{c}{g(s)}\right)^{1/2}\right]}.$$

Here

(37) $$|g(s)-g(\theta)|=4\left|\sin\left(\frac{s-\theta}{2}\right)\sin\left(\frac{s+\theta}{2}\right)\right|$$

$$=2\,|e^{is}-e^{i\theta}|\left|\sin\frac{s}{2}\cos\frac{\theta}{2}+\cos\frac{s}{2}\sin\frac{\theta}{2}\right|$$

$$\leqslant 4\,|e^{is}-e^{i\theta}|\min\left\{\sin\frac{s}{2},\cos\frac{\alpha}{2}\right\}.$$

(We have used the fact that $s, \theta \in [\alpha, 2\pi - \alpha]$ and also (36)). Also,

$$|c| \leqslant 4\left(\sin\frac{\alpha}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2$$
$$\leqslant 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2 \leqslant \pi^2 g(\theta).$$

Then

$$n\left|\frac{\varepsilon(e^{i\theta}) - \varepsilon(e^{is})}{e^{i\theta} - e^{is}}\right| \leqslant \frac{4\pi^2 \min\left\{\sin\frac{s}{2}, \cos\frac{\alpha}{2}\right\}}{g(s)\left(1 + \frac{c}{g(s)}\right)^{1/2}}$$
$$= \frac{4\pi^2 \min\left\{\sin\frac{s}{2}, \cos\frac{\alpha}{2}\right\}}{(f(s)\, g(s))^{1/2}}.$$

We now consider two subcases:

Case I: $\alpha \leqslant \frac{\pi}{2}$. Here we use

$$f(s)^{1/2} \geqslant \frac{\pi - \alpha}{n} \geqslant \frac{\pi}{2n},$$
$$g(s)^{1/2} \geqslant 2\left|\sin\frac{s}{2}\right|$$

to deduce

$$\left|\frac{\varepsilon(e^{i\theta}) - \varepsilon(e^{is})}{e^{i\theta} - e^{is}}\right| \leqslant 4\pi < 14.$$

Case II: $\alpha > \frac{\pi}{2}$. Here we use

$$f(s)^{1/2} \geqslant \frac{\pi - \alpha}{n} \geqslant \frac{2\cos\frac{\alpha}{2}}{n},$$

by (35), and also

$$g(s)^{1/2} \geqslant 2\left|\sin\frac{s}{2}\right| \geqslant 2\sin\frac{\pi}{4}$$

to deduce

$$\left|\frac{\varepsilon(e^{i\theta})-\varepsilon(e^{is})}{e^{i\theta}-e^{is}}\right| \leqslant \frac{\pi^2}{\sin\frac{\pi}{4}} < 14.$$

(c) This is an immediate consequence of (b).

(d) Our restrictions on s, θ give

$$\left|\frac{s-\theta}{2}\right| \in [0, \pi].$$

Then

$$0 \leqslant \sin\left|\frac{s-\theta}{2}\right| = \frac{1}{2}|e^{is}-e^{i\theta}| \leqslant \frac{r}{2}$$

$$\Rightarrow \left|\frac{s-\theta}{2}\right| \in \left[0, \arcsin\frac{r}{2}\right] \cup \left[\pi - \arcsin\frac{r}{2}, \pi\right].$$

It follows that s can lie in a set of linear Lebesgue measure at most $8 \arcsin \frac{r}{2}$. The inequality

$$\arcsin u \leqslant \frac{\pi}{2} u, \qquad u \in [0, 1]$$

then gives the result.

We next discuss the growth of the conformal map

$$\Psi(z) = \frac{1}{2\cos\frac{\alpha}{2}}\left[z+1+\sqrt{R(z)}\right], \tag{38}$$

mapping $\mathbb{C}\setminus\Delta$ onto $\{w: |w|>1\}$. The proof here is more complex than that in [7], because of the more difficult choice of $\varepsilon(z)$.

LEMMA 3.2. *Let $\ell \geqslant 1$. For $a \in \Delta$ and $z \in \mathbb{C}$ such that*

$$|z-a| \leqslant \varepsilon(a)/100, \tag{39}$$

we have

$$|\Psi(z)|^{n+\ell} \leqslant C_0. \tag{40}$$

Here C_0 depends on ℓ, but is independent of n, α, z.

Proof. We shall assume that $|z| \geqslant 1$. The case $|z| < 1$ is similar. Let us write

(41) $$z = te^{i\theta} = e^{i\xi} \qquad \text{where} \quad \xi = \theta - i \log t$$

and set

$$v := e^{i\theta}.$$

We consider two subcases.

(A) Suppose that $v \in \Delta$.
We shall show that for some numerical constant C_1,

(42) $$|\Psi(z) - \Psi(v)| = |\Psi(z) - \Psi(v)_-| \leqslant \frac{C_1}{n+1}.$$

Then as $|\Psi(v)| = 1$, we obtain

$$|\Psi(z)|^{n+\ell} \leqslant \left(1 + \frac{C_1}{n+1}\right)^{n+\ell} \leqslant C_0.$$

First we see that

(43) $$|\Psi(z) - \Psi(v)| \leqslant \frac{|z-v|}{2 \cos \alpha/2} + \frac{|\sqrt{R(z)} - \sqrt{R(v)}|}{2 \cos \alpha/2} =: T_1 + T_2.$$

Here

$$T_1 = \frac{|z-v|}{2 \cos \alpha/2} \leqslant \frac{|z-a|}{2 \cos \alpha/2} \leqslant \frac{\varepsilon(a)}{200 \cos \frac{\alpha}{2}} \leqslant \frac{1}{n+1},$$

by Lemma 3.1(a). We turn to the more difficult estimation of

(44) $$T_2 := \frac{|\sqrt{R(z)} - \sqrt{R(v)}|}{2 \cos \alpha/2}.$$

We see from (10) that

$$\begin{aligned} R(v) - R(z) &= (v^2 - 2(\cos \alpha)\, v + 1) - (z^2 - 2(\cos \alpha)\, z + 1) \\ &= (v - z)(z - v + 2(v - \cos \alpha)) \\ &= -(v-z)^2 + 2(v-z)(\cos \theta - \cos \alpha) + 2i(\sin \theta)(v - z). \end{aligned}$$

Then

$$|R(z)-R(v)| \leqslant |v-z|\left(|v-z|+4\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}\right)+2\,|\sin\theta|\right) \tag{45}$$
$$= |v-z|(|v-z|+|R(v)|+2\,|\sin\theta|);$$

see (27). We now consider two subcases:

Case I: $|R(v)| \leqslant (\frac{\pi-\alpha}{n})^2$. Then as

$$|a-v| \leqslant |a-z| \leqslant \varepsilon(a)/100,$$

Lemma 3.1(c), followed by (11), gives

$$\varepsilon(a) \leqslant 2\varepsilon(v) \leqslant \frac{2\sqrt{2}\left(\frac{\pi-\alpha}{n}\right)}{n\left(\left(\sin\frac{\theta}{2}\right)^2+\left(\frac{1}{n}\right)^2\right)^{1/2}} \leqslant 2\sqrt{2}\,\frac{\pi-\alpha}{n}\min\left\{1, \frac{1}{n\left|\sin\frac{\theta}{2}\right|}\right\}.$$

Also,

$$|v-z| \leqslant |a-z| \leqslant \frac{\varepsilon(a)}{100} \leqslant C\,\frac{\pi-\alpha}{n}.$$

Then (45) and our assumption on $R(v)$ give

$$|R(z)-R(v)| \leqslant C\left\{\left(\frac{\pi-\alpha}{n}\right)^2+\left(\frac{\pi-\alpha}{n}\right)^2+\varepsilon(a)\,2\left|\sin\frac{\theta}{2}\right|\left|\cos\frac{\theta}{2}\right|\right\}$$
$$\leqslant C\left\{\left(\frac{\pi-\alpha}{n}\right)^2+\frac{\pi-\alpha}{n^2\left|\sin\frac{\theta}{2}\right|}\left|\sin\frac{\theta}{2}\right|\left|\cos\frac{\alpha}{2}\right|\right\}$$
$$\leqslant C\left(\frac{\pi-\alpha}{n}\right)^2;$$

recall also that $\cos\frac{\theta}{2} \leqslant \cos\frac{\alpha}{2}$. Hence

$$|R(z)| \leqslant C\left(\frac{\pi-\alpha}{n}\right)^2.$$

Then we see from (44) that

$$T_2 \leqslant \frac{C}{n}. \tag{46}$$

Case II: $|R(v)| > (\frac{\pi-\alpha}{n})^2$. As above, Lemma 3.1(c) gives

(47)
$$\varepsilon(a) \leqslant 2\varepsilon(v) \leqslant \frac{2\sqrt{2}\,|R(v)|^{1/2}}{n\left(\left(\sin\frac{\theta}{2}\right)^2+\left(\frac{1}{n}\right)^2\right)^{1/2}} \leqslant 2\sqrt{2}\,|R(v)|^{1/2}\min\left\{1, \frac{1}{n\left|\sin\frac{\theta}{2}\right|}\right\}.$$

Then (45) and the fact that $|R(v)| \leqslant 4$ give

$$\begin{aligned}|R(z)-R(v)| &\leqslant \frac{\varepsilon(a)}{100}\left(\frac{\varepsilon(a)}{100}+|R(v)|+2\left|\sin\frac{\theta}{2}\right|\left|\cos\frac{\theta}{2}\right|\right)\\ &\leqslant \frac{8}{10,000}|R(v)|+\frac{4\sqrt{2}}{100}|R(v)|+\frac{4\sqrt{2}}{100}\frac{|R(v)|^{1/2}}{n}\cos\frac{\alpha}{2}.\end{aligned}$$

But

$$|R(v)|^{1/2} > \frac{\pi-\alpha}{n} \geqslant 2\frac{\cos\frac{\alpha}{2}}{n},$$

so

$$|R(z)-R(v)| \leqslant \frac{1}{4}|R(v)|.$$

It then follows that for some numerical constant C,

$$|\sqrt{R(v)}-\sqrt{R(z)}| \leqslant C\frac{|R(v)-R(z)|}{\sqrt{|R(v)|}}.$$

(See the proof of Lemma 3.2 in [7] for a detailed justification of this inequality.) Then from (44) and (45),

(48)
$$\begin{aligned}T_2 &\leqslant C\left\{\frac{|v-z|^2}{\cos\frac{\alpha}{2}|R(v)|^{1/2}}+\frac{|v-z|\,|R(v)|^{1/2}}{\cos\frac{\alpha}{2}}+\frac{|\sin\theta|\,|v-z|}{|R(v)|^{1/2}\cos\frac{\alpha}{2}}\right\}\\ &=: C\{T_{21}+T_{22}+T_{23}\}.\end{aligned}$$

Here from (31), (47),

$$T_{21} = \frac{|v-z|^2}{\cos\frac{\alpha}{2}\,|R(v)|^{1/2}} \leqslant \frac{\varepsilon(a)^2}{\cos\frac{\alpha}{2}\,|R(v)|^{1/2}}$$

$$\leqslant \frac{\left(6\,\frac{\cos\frac{\alpha}{2}}{n}\right)(2\sqrt{2}\,|R(v)|^{1/2})}{\cos\frac{\alpha}{2}\,|R(v)|^{1/2}} = \frac{12\sqrt{2}}{n}.$$

Next,

$$T_{22} = \frac{|v-z|\,|R(v)|^{1/2}}{\cos\frac{\alpha}{2}} \leqslant \frac{\varepsilon(a)\cdot 2}{\cos\frac{\alpha}{2}} \leqslant \frac{12}{n},$$

by (31). Finally,

$$T_{23} = \frac{|\sin\theta|\,|v-z|}{|R(v)|^{1/2}\cos\frac{\alpha}{2}} \leqslant \frac{2\left|\sin\frac{\theta}{2}\right|\left(\cos\frac{\alpha}{2}\right)\varepsilon(a)}{|R(v)|^{1/2}\cos\frac{\alpha}{2}}$$

$$\leqslant \frac{4\sqrt{2}}{n},$$

by (47). Then these estimates and (48) give

$$T_2 \leqslant C/n,$$

and then we have the desired inequality (42).

(B) Suppose that $v \notin \Delta$.

Then $\theta \in [0, \alpha)$ or $\theta \in (2\pi - \alpha, 2\pi]$. We assume the former. We also assume that $a = e^{is}$ with $s \in [\alpha, \pi]$ (the case $s \in (\pi, 2\pi - \alpha]$ is easier). Then

$$|\Psi(z) - \Psi(e^{i\alpha})| = \frac{1}{2\cos\frac{\alpha}{2}}\,|z - e^{i\alpha} + \sqrt{R(z)}| \tag{49}$$

$$\leqslant \frac{|z - e^{i\alpha}|}{2\cos\frac{\alpha}{2}} + \frac{|R(z)|^{1/2}}{2\cos\frac{\alpha}{2}}.$$

Here, as above,

$$|z-e^{i\alpha}| \leqslant |z-a|+|a-e^{i\alpha}| \leqslant \frac{\varepsilon(a)}{50},$$

so from Lemma 3.1(c), and then (11),

$$\varepsilon(a) \leqslant 2\varepsilon(e^{i\alpha}) = \frac{2\left(\frac{\pi-\alpha}{n}\right)}{n\left(4\left(\sin\frac{\alpha}{2}\right)^2+\frac{1}{n^2}\right)^{1/2}} \leqslant 2\pi\,\frac{\cos\frac{\alpha}{2}}{n}\min\left\{1, \frac{1}{n\left|\sin\frac{\alpha}{2}\right|}\right\}. \tag{50}$$

Then from (31),

$$\frac{|z-e^{i\alpha}|}{2\cos\frac{\alpha}{2}} \leqslant \frac{\varepsilon(a)}{100\cos\frac{\alpha}{2}} \leqslant \frac{6}{n}. \tag{51}$$

Next,

$$\begin{aligned}
|R(z)| &= |z-e^{i\alpha}|\,|z-e^{-i\alpha}| \\
&\leqslant |z-e^{i\alpha}|\,(|z-e^{i\alpha}|+2\sin\alpha) \\
&\leqslant \varepsilon(a)^2+\frac{\varepsilon(e^{i\alpha})}{25}\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\
&\leqslant C\left(\frac{\cos\frac{\alpha}{2}}{n}\right)^2+C\,\frac{\pi-\alpha}{n^2}\cos\frac{\alpha}{2} \\
&\leqslant C\left(\frac{\cos\frac{\alpha}{2}}{n}\right)^2.
\end{aligned}$$

Here we have used (50). This last inequality and (49), (51) give

$$|\Psi(z)| \leqslant |\Psi(e^{i\alpha})|+\frac{C}{n} = 1+\frac{C}{n},$$

and again (42) follows. ∎

We next estimate the norms of the Carleson measures σ^+, $\sigma^\#$ defined by (14) and (17)–(18). Recall that the Carleson norm $N(\mu)$ of a measure μ with support in the unit ball is the least A such that

$$\mu(S) \leqslant Ah \tag{52}$$

for every $0 < h < 1$ and for every sector

$$S := \{re^{i\theta} : r \in [1-h, 1]; |\theta-\theta_0| \leqslant h\}. \tag{53}$$

LEMMA 3.3. (a)

$$N(\sigma^+) \leqslant c_1. \tag{54}$$

(b)

$$N(\sigma^\#) \leqslant c_2. \tag{55}$$

Proof. (a) We proceed much as in [7], [8], or [10]. Let S be the sector (53) and let γ be a circle centre a, radius $\frac{\varepsilon(a)}{100} > 0$. A necessary condition for γ to intersect S is that

$$|a-e^{i\theta_0}| \leqslant \frac{\varepsilon(a)}{100} + h.$$

(Note that each point of S that is on the unit circle is at most h in distance from $e^{i\theta_0}$.) Using Lemma 3.1(b), we continue this as

$$|a-e^{i\theta_0}| \leqslant \frac{\varepsilon(e^{i\theta_0})}{100} + \frac{14}{100}|a-e^{i\theta_0}| + h \tag{56}$$

$$\Rightarrow |a-e^{i\theta_0}| \leqslant \frac{\varepsilon(e^{i\theta_0})}{86} + 2h =: \lambda.$$

Next $\gamma \cap S$ consists of at most three arcs (draw a picture!) and as each such arc is convex, it has length at most $4h$. Therefore the total angular measure of $\gamma \cap S$ is at most $12h/(\varepsilon(a)/100)$. It also obviously does not exceed 2π. Thus if χ_S denotes the characteristic function of S,

$$\int_{-\pi}^{\pi} \chi_S(a+\varepsilon(a)\, e^{i\theta})\, d\theta \leqslant \min\left\{2\pi, \frac{1200h}{\varepsilon(a)}\right\}.$$

Then from (14) and (17), we see that

$$(57)\quad \sigma^+(S) \leqslant \sigma(S)$$
$$\leqslant \int_{[\alpha,\, 2\pi-\alpha] \cap \{s:\, |e^{is}-e^{i\theta_0}| \leqslant \lambda\}} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_S\left(e^{is} + \frac{\varepsilon(e^{is})}{100}\, e^{i\theta} \right) d\theta \right] ds$$
$$\leqslant C_1 \int_{[\alpha,\, 2\pi-\alpha] \cap \{s:\, |e^{is}-e^{i\theta_0}| \leqslant \lambda\}} \min\left\{1, \frac{h}{\varepsilon(e^{is})}\right\} ds.$$

Here C_1 is a numerical constant. We now consider two subcases:

(I) $h \leqslant \varepsilon(e^{i\theta_0})/100$. In this case,

$$\lambda < \frac{\varepsilon(e^{i\theta_0})}{25} < 1;$$

recall (31). Then Lemma 3.1(d) shows that s in the integral in (57) lies in a set of linear Lebesgue measure at most

$$2\pi \cdot \frac{\varepsilon(e^{i\theta_0})}{25}.$$

Also Lemma 3.1(c) gives

$$\varepsilon(e^{is}) \geqslant \tfrac{1}{2}\, \varepsilon(e^{i\theta_0}).$$

So (57) becomes

$$\sigma^+(S) \leqslant \sigma(S) \leqslant C_1 \left(2\pi \cdot \frac{\varepsilon(e^{i\theta_0})}{25} \right) \left(2 \frac{h}{\varepsilon(e^{i\theta_0})} \right) = C_2 h.$$

(II) $h > \varepsilon(e^{i\theta_0})/100$. In this case $\lambda < 4h$. If $h < \frac{1}{2}$, we obtain from Lemma 3.1(d) that s in the integral in (57) lies in a set of linear Lebesgue measure at most $2\pi \cdot 4h$. Then (57) becomes

$$\sigma^+(S) \leqslant \sigma(S) \leqslant C_1 (2\pi \cdot 4h) = C_2 h.$$

If $h > \frac{1}{2}$, it is easier to use

$$\sigma^+(S) \leqslant \sigma(S) \leqslant \sigma(\mathbb{C}) \leqslant 2\pi \leqslant 4\pi h.$$

In summary, we have proved that

$$N(\sigma^+) = \sup_{S,\,h} \frac{\sigma^+(S)}{h} \leqslant C_3,$$

where C_3 is independent of n, α, β. (It is also independent of p.)

(b) Recall that if S is the sector (53), then

$$\sigma^\#(S) = \sigma^-(1/S) \leqslant \sigma(1/S),$$

where

$$1/S = \left\{ re^{i\theta} : r \in \left[1, \frac{1}{1-h}\right];\ |\theta + \theta_0| \leqslant h \right\}.$$

For small h, say for $h \in [0, 1/2]$, so that

$$\frac{1}{1-h} \leqslant 1 + 2h,$$

we see that exact same argument as in (a) gives

$$\sigma^\#(S) \leqslant \sigma(1/S) \leqslant C_4 h.$$

When $h \geqslant 1/2$, it is easier to use

$$\sigma^\#(S)/h \leqslant 2\sigma^\#(\mathbb{C}) \leqslant 2\sigma(\mathbb{C}) \leqslant 4\pi. \quad \blacksquare$$

4. THE PROOF OF THEOREM 1.2

We deduce Theorem 1.2 from Theorem 1.3 as follows: if s_n is a trigonometric polynomial of degree $\leqslant n$, we may write

$$s_n(\theta) = e^{-in\theta} P(e^{i\theta}),$$

where P is an algebraic polynomial of degree $\leqslant 2n$. Then

$$|s_n'(\theta)|\ \varepsilon_{2n}(\varepsilon^{i\theta}) \leqslant n\ |P(e^{i\theta})|\ \varepsilon_{2n}(e^{i\theta}) + |P'(e^{i\theta})|\ \varepsilon_{2n}(\varepsilon^{i\theta}).$$

Moreover,

$$|e^{i\theta} - e^{i\alpha}|\ |e^{i\theta} - e^{i\beta}| = 4\left|\sin\left(\frac{\theta - \alpha}{2}\right)\right| \left|\sin\left(\frac{\theta - \beta}{2}\right)\right|,$$

and

$$|e^{i\theta}+e^{i\frac{\alpha+\beta}{2}}|^2=4\left(\cos\left(\theta-\frac{\alpha+\beta}{2}\right)\right)^2.$$

These last three relations, the fact that $n\varepsilon_{2n}(e^{i\theta})$ is bounded independent of n, θ, α, β, and Theorem 1.3 easily imply (4).

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